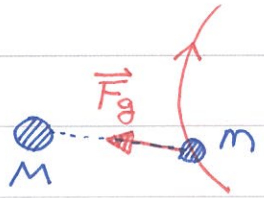
HH0096 Gravity and Spacetime Curvature.

Consider a particle of mass $m \ll M$ moving under the gravitational influence of another much heavier particle M. Newton's description can be separated into 2 steps.



① Force: $\vec{F_g} = -\frac{GMm}{r^2}\hat{r}$

② EOM: $\vec{F_g} = m\frac{d^2\vec{r}}{dt^2}$

Combine ①+② and we can find the trajectory

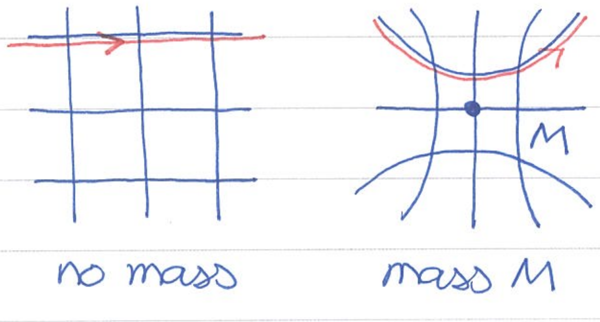
But, let's take a closer look, $-\frac{GM\cancel{m}}{r^2}\hat{r} = \cancel{m}\frac{d^2\vec{r}}{dt^2}$.

The mass m drops out ...

$$\boxed{-\frac{GM}{r^2}\hat{r} = \frac{d^2\vec{r}}{dt^2}}$$

It is amusing that the trajectory of the particle is independent of its mass.

The above result encourages Einstein to view "the trajectory" as the cosequence of "curved spacetime".



geodesic (straight line in curved 4 dimensinal spacetime)

Einstein's view can also be separated into 2 steps.

① Mass produces spacetime curvature:

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}}$$

Some explanations are in order....

$T_{\mu\nu}$: energy-stress tensor

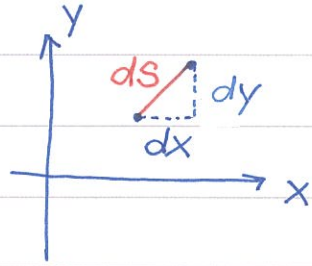
basically, just "masses".

② The particle just goes "straight" in the curved spacetime, i.e. the geodesic !

$R_{\mu\nu}$, R are curvature tensors – 2^{nd} derivatives of the metric tensor $g_{\mu\nu}$.

how spacetime is curved

① Metric tensor in 2D. Consider the flat x-y plane without any curvature.



$$(ds)^2 = (dx)^2 + (dy)^2$$

In general, the distance can take the form

$$\boxed{(ds)^2 \equiv \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\, dx^i dx^j}$$

$dx^1 = dx$
$dx^2 = dy$

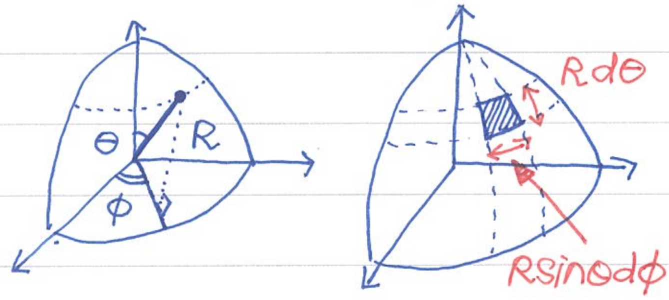
It shall be easy to see that $g_{11} = g_{22} = 1$, $g_{12} = g_{21} = 0$.

Or, in matrix form:

$$\boxed{g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

The metric tensor is trivial.

Now consider the more challenging problem – the curved 2D plane on a sphere. We use two angles (ϕ, θ) as the coordinates and try to work out the metric tensor g_{ij}.



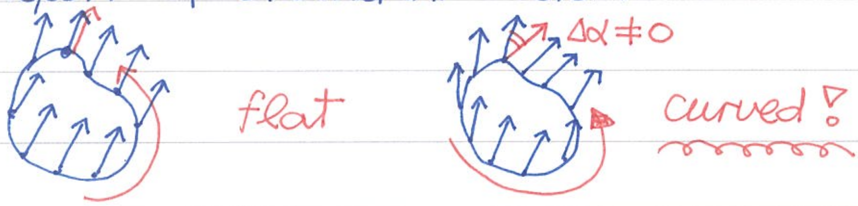
$$(ds)^2 = R^2 \sin^2\theta\, (d\phi)^2 + R^2 (d\theta)^2$$

Follow the same steps as before, the metric tensor can be extracted,

$$\rightarrow \boxed{g_{ij} = \begin{pmatrix} R^2 \sin^2\theta & 0 \\ 0 & R^2 \end{pmatrix}}$$

This metric tensor is non-trivial. turns out to be curved.

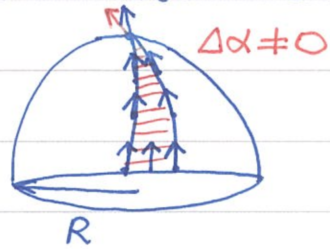
From the above metric tensor, one can compute the corresponding curvature. But, one needs to learn more math.... A simpler (and intuitive) way to estimate curvature C is by "parallel transport" of a constant vector.



The vector does not go back to the same direction.

The curvature C is defined as the ratio of the tilted angle $\Delta\alpha$ and the enclosed area A.

$$\boxed{C = \lim_{A\to 0} \frac{\Delta\alpha}{A}}$$

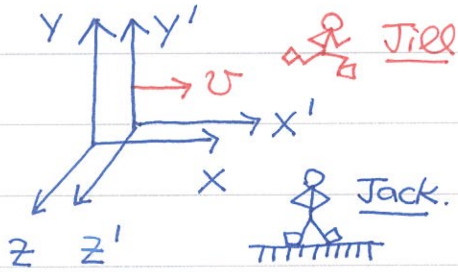


Consider a closed contour on the sphere of radius R as shown here.

$$A = \left(\frac{\Delta\alpha}{2\pi}\right)\cdot 2\pi R^2 = R^2 \Delta\alpha \longrightarrow \boxed{C = \lim_{A\to 0}\frac{\Delta\alpha}{A} = \frac{1}{R^2} \neq 0}$$

Note that, in principle, C can be computed directly from g_{ij}.

⊘ Metric tensor in 4D. Now we are ready to compute the metric tensor $g_{\mu\nu}$ for the 4 dimensional spacetime. Let us start from the inertial frames, which are related by the Lorentz transformation:



$$\boxed{\begin{aligned} \Delta x' &= \gamma\Delta x - \frac{\gamma v}{c}(c\Delta t) \\ c\Delta t' &= \gamma(c\Delta t) - \frac{\gamma v}{c}\Delta x \end{aligned}}$$

Lorentz trans.

And, $\Delta y' = \Delta y$, $\Delta z' = \Delta z$.

Let us try to look for "something" invariant in both Jack and Jill's frames.

$$-(c\Delta t')^2 + (\Delta x')^2 = -\left[\gamma(c\Delta t) - \frac{\gamma v}{c}\Delta x\right]^2 + \left[\gamma\Delta x - \frac{\gamma v}{c}(c\Delta t)\right]^2$$

The algebra is straightforward

$$-(c\Delta t')^2 + (\Delta x')^2 = -\gamma^2\left(1-\frac{v^2}{c^2}\right)(c\Delta t)^2 + \gamma^2\left(1-\frac{v^2}{c^2}\right)(\Delta x)^2$$

But $\gamma^2(1 - v^2/c^2) = 1$, the above relation simplified:

$$\underbrace{-(c\Delta t')^2 + (\Delta x')^2 = -(c\Delta t)^2 + (\Delta x)^2}$$

Something invariant.

Introduce the "spacetime distance" ds as the following

$$\boxed{\begin{aligned}(ds)^2 &= -(c\,dt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \\ &= -(c\,dt')^2 + (dx')^2 + (dy')^2 + (dz')^2\end{aligned}}$$

Clearly, $(ds)^2$ is invariant in both Jack and Jill's frames. The extracted metric tensor $g_{\mu\nu}$ is

$$\boxed{g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

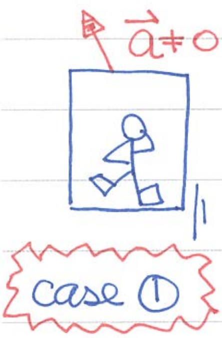
$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ 4 dimensional spacetime

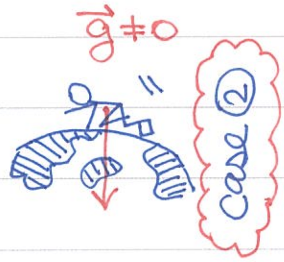
The metric tensor for inertial frames is quite simple oooo

In fact, starting from the above $g_{\mu\nu}$, one can derive the constancy of c and thus the Lorentz transformation ☺

⑪ Curved spacetime. The simple metric $g_{\mu\nu}$ for inertial frames is gone if ① the frame is accelerating ($\vec{a} \neq 0$)

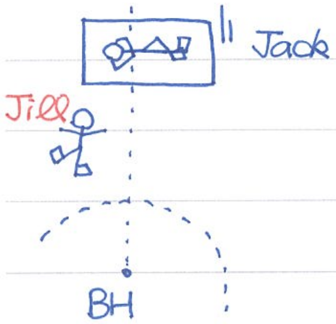
② gravity is present ($\vec{g} \neq 0$).





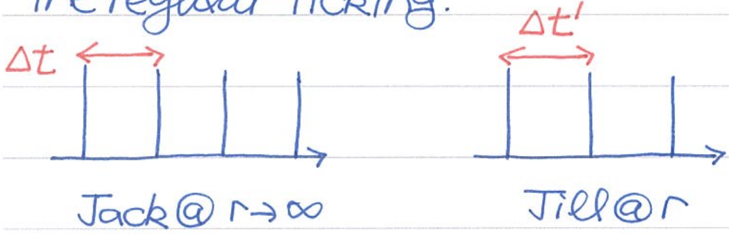
OK, bad news. But Einstein tells us that you cannot distinguish the effects due to $\vec{a}$ or $\vec{g}$ (as explained in previous lectures).

It is then fun to consider the Jack and Jill problem outside the event horizon of a black hole. Suppose Jack was initially at rest and very far away from the black hole. He is in an inertial frame and only needs special relativity. Jill is also at rest but closer (but not too close) to the BH with distance r.



The gravitational potential $\Phi = -\frac{GM}{r} \neq 0$. Thus, Jill needs to learn general relativity, i.e. the horrible looking tensor equations ....

Jack and Jill carry the same clock and observe the regular ticking.



$$\boxed{\Delta t' = \left(1 + \frac{GM}{c^2 r}\right) \Delta t}$$

We would like to show the above time dilation.

Here is Einstein's neat trick. Let Jack falls freely from infinity so that the effects due to acceleration and gravity cancel. Jack remains in the same inertial frame and his clock ticks at time interval Δt. When falling to Jill's location, the relative velocity v is

$$\frac{mc^2}{\sqrt{1 - v^2/c^2}} + m\Phi = mc^2 + 0$$

$$\Rightarrow \boxed{\frac{1}{\sqrt{1 - v^2/c^2}} = 1 - \frac{\Phi}{c^2}} \qquad \text{Here } \Phi = -\frac{GM}{r},$$

Jack finds Jill is moving at speed v and concludes that

$$\boxed{\Delta t' = \frac{1}{\sqrt{1 - v^2/c^2}} \Delta t = \left(1 - \frac{\Phi}{c^2}\right) \Delta t = \left(1 + \frac{GM}{c^2 r}\right) \Delta t}$$

It is important to emphasize that Jack can use special relativity to make the above claim because he is in the inertial frame. On the other hand, if Jill tries to compare her $\Delta t'$ with Jack's Δt, she needs to use general relativity.

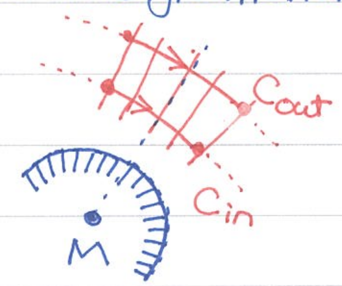
Similarly, Jack will find the length contraction. assuming $\frac{GM}{c^2 r} \ll 1$.

$$\boxed{\Delta x' = \sqrt{1 - v^2/c^2}\, \Delta x = \left(1 + \frac{GM}{c^2 r}\right)^{-1} \Delta x \approx \left(1 - \frac{GM}{c^2 r}\right) \Delta x}$$

The velocity transformation can be derived,

$$\boxed{v' = \frac{\Delta x'}{\Delta t'} = \left(1+\frac{GM}{c^2 r}\right)^{-2} \frac{\Delta x}{\Delta t} \approx \left(1-\frac{2GM}{c^2 r}\right) v}$$

Apply the above relation to the speed of light, $c' = (1 - 2GM/c^2 r)\, c < c$ — light slows down in the gravitational field! Consider two light beams passing at different distances, $r_{out} > r_{in}$



$\rightarrow$ $c_{out} > c_{in}$

The wave fronts get deflected due to velocity mismatch!

Thus, just like refraction, light passing by massive stars will get deflected a bit.

If one sits down and work out all results from general relativity, you will get the following:

$$\boxed{\begin{aligned} \Delta x' &= \sqrt{1-\frac{2GM}{rc^2}}\, \Delta x \\ \Delta t' &= \frac{1}{\sqrt{1-\frac{2GM}{rc^2}}}\, \Delta t \end{aligned}}$$

$R_S = \frac{2GM}{c^2}$

$\rightarrow$ $c' = \left(1 - \frac{2GM}{rc^2}\right) c$

When $r \to R_S^+$, c' slows down to 0!

One can see that the black hole distorts the spacetime strongly so that $c' = 0$ right at the event horizon ☺

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